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Global solvability and asymptotic behavior for a nonlinear coupled system of viscoelastic waves with memory in a noncylindrical domain

M.L. Santos *, M.P.C. Rocha, P.L.O. Braga

Departamento de Matemática, Universidade Federal do Pará (UFPA), Campus Universitário do Guamá, Rua Augusto Corrêa 01, CEP 66075-110, Belém-Pa., Brazil
Instituto de Estudos Superiores da Amazônia (IESAM), Av. Gov. José Malcher 1148, CEP 66.055-260, Belém-Pa., Brazil

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Abstract

In this paper we prove the exponential decay in the case $n > 2$, as time goes to infinity, of regular solutions for a nonlinear coupled system of wave equations with memory and weak damping

$$u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)\,ds + \alpha u_t + h(u-v) = 0 \quad \text{in } \hat{Q},$$

$$v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)\,ds + \alpha v_t - h(u-v) = 0 \quad \text{in } \hat{Q},$$

in a noncylindrical domains $\hat{Q}$ of $\Re^{n+1}$ $(n \geqslant 1)$ under suitable hypotheses on the scalar functions h, g_1 and g_2, and where α is a positive constant. We show that such dissipation is strong enough to produce uniform rate decay. Besides, the coupled is nonlinear which brings up some additional difficulties, which make the problem interesting. We establish existence and uniqueness of regular solutions for any $n \geqslant 1$.

Keywords: Nonlinear coupled system; Wave equations; Noncylindrical domain; Global solution; Exponential decay

* Corresponding author.
E-mail address: ls@ufpa.br (M.L. Santos).

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1. Introduction

Let Ω be an open bounded domains of $\mathbb{R}^n$ with boundary of class C^2, which, without loss of generality, can be assumed containing the origin of $\mathbb{R}^n$ and $\gamma : [0, \infty[\to \mathbb{R}$ a continuously differentiable function. See hypotheses (1.13)–(1.15) on γ. Let us consider the family of subdomains $\{\Omega_t\}_{0\leqslant t<\infty}$ of $\mathbb{R}^n$ given by

$$\Omega_t = T(\Omega), \quad T : y \in \Omega \mapsto x = \gamma(t)y$$

whose boundaries are denoted by Γ_t, and $\hat{Q}$ the noncylindrical domains of $\mathbb{R}^{n+1}$,

$$\hat{Q} = \bigcup_{0\leqslant t<\infty} \Omega_t \times \{t\}$$

with lateral boundary

$$\hat{\Sigma} = \bigcup_{0\leqslant t<\infty} \Gamma_t \times \{t\}.$$

Let us consider the Hilbert space $L^2(\Omega)$ endowed with the inner product

$$(u, v) = \int_\Omega u(x)v(x)\,dx$$

and corresponding norm

$$\|u\|^2_{L^2(\Omega)} = (u, u).$$

We also consider the Sobolev space $H^1(\Omega)$ endowed with the scalar product

$$(u, v)_{H^1(\Omega)} = (u, v) + (\nabla u, \nabla v).$$

We define the subspace of $H^1(\Omega)$, denoted by $H^1_0(\Omega)$, as the closure of $C^\infty_0(\Omega)$ in the strong topology of $H^1(\Omega)$. By $H^{-1}(\Omega)$ we denote the dual space of $H^1_0(\Omega)$. This space endowed with the norm induced by the scalar product

$$((u, v))_{H^1_0(\Omega)} = (\nabla u, \nabla v)$$

is, owing to the Poincaré inequality

$$\|u\|^2_{L^2(\Omega)} \leqslant C\|\nabla u\|^2_{L^2(\Omega)},$$

a Hilbert space. We define for all $1 \leqslant p < \infty$,

$$\|u\|^p_{L^p(\Omega)} = \int_\Omega \left|u(x)\right|^p dx,$$

and if $p = \infty$,

$$\|u\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} \operatorname{ess}\left|u(x)\right|.$$

In this work we study the existence and uniqueness of strong solutions as well as the exponential decay of the energy to the nonlinear coupled system of wave equations with memory and weak damping given by

$$u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)\,ds + \alpha u_t + h(u-v) = 0 \quad \text{in } \hat{Q}, \tag{1.1}$$

$$v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)\,ds + \alpha v_t - h(u-v) = 0 \quad \text{in } \hat{Q}, \tag{1.2}$$

$$u = v = 0 \quad \text{on } \hat{\Sigma}, \tag{1.3}$$

$$\big(u(x,0), v(x,0)\big) = \big(u_0(x), v_0(x)\big), \quad \big(u_t(x,0), v_t(x,0)\big) = \big(u_1(x), v_1(x)\big) \quad \text{in } \Omega_0, \tag{1.4}$$

where u, v are the transverse displacements and α is a positive constant.

The method we use to prove the result of existence and uniqueness is based on transforming our problem into another initial boundary value problem defined over a cylindrical domain whose sections are not time-dependent. This is done using a suitable change of variable. Then we show the existence and uniqueness for this new problem. Our existence result on noncylindrical domains will follows using the inverse transformation. That is, using the diffeomorphism $\tau : \hat{Q} \to Q$ defined by

$$\tau : \hat{Q} \to Q, \quad (x,t) \in \Omega_t \mapsto (y,t) = \left(\frac{x}{\gamma(t)}, t\right) \tag{1.5}$$

and $\tau^{-1} : Q \to \hat{Q}$ defined by

$$\tau^{-1}(y,t) = (x,t) = \big(\gamma(t)y, t\big). \tag{1.6}$$

Denoting by φ and ψ the functions

$$\varphi(y,t) = u \circ \tau^{-1}(y,t) = u\big(\gamma(t)y, t\big), \qquad \psi(y,t) = v \circ \tau^{-1}(y,t) = v\big(\gamma(t)y, t\big) \tag{1.7}$$

the initial boundary value problem (1.1)–(1.4) becomes

$$\begin{aligned} &\varphi_{tt} - \gamma^{-2}\Delta\varphi + \int_0^t g_1(t-s)\gamma^{-2}(s)\Delta\varphi(s)\,ds + \alpha\varphi_t \\ &\quad + h(\varphi - \psi) - A(t)\varphi + a_1 \cdot \nabla\partial_t\varphi + a_2 \cdot \nabla\varphi = 0 \quad \text{in } Q, \end{aligned} \tag{1.8}$$

$$\begin{aligned} &\psi_{tt} - \gamma^{-2}\Delta\psi + \int_0^t g_2(t-s)\gamma^{-2}(s)\Delta\psi(s)\,ds + \alpha\psi_t \\ &\quad - h(\varphi - \psi) - A(t)\psi + a_1 \cdot \nabla\partial_t\psi + a_2 \cdot \nabla\psi = 0 \quad \text{in } Q, \end{aligned} \tag{1.9}$$

$$\varphi|_\Gamma = \psi|_\Gamma = 0, \tag{1.10}$$

$$(\varphi|_{t=0}, \psi|_{t=0}) = (\varphi_0, \psi_0), \quad (\varphi_t|_{t=0}, \psi_t|_{t=0}) = (\varphi_1, \psi_1) \quad \text{in } \Omega, \tag{1.11}$$

where

$$A(t)\varphi = \sum_{i,j=1}^n \partial_{y_i}(a_{ij}\partial_{y_j}\varphi), \qquad A(t)\psi = \sum_{i,j=1}^n \partial_{y_i}(a_{ij}\partial_{y_j}\psi)$$

and

$$\begin{cases} a_{ij}(y,t) = -(\gamma'\gamma^{-1})^2 y_i y_j \quad (i,j=1,\dots,n), \\ a_1(y,t) = -2\gamma'\gamma^{-1} y, \\ a_2(y,t) = -\gamma^{-2} y(\gamma''\gamma + \gamma'(\alpha\gamma + (n-1)\gamma')). \end{cases} \tag{1.12}$$

To show the existence of strong solutions we will use the following hypotheses:

$$\gamma' \leqslant 0 \quad \text{if } n > 2, \qquad \gamma' \geqslant 0 \quad \text{if } n \leqslant 2, \tag{1.13}$$

$$\gamma(\cdot) \in L^\infty(0,\infty), \qquad \inf_{0 \leqslant t < \infty} \gamma(t) = \gamma_0 > 0, \tag{1.14}$$

$$\gamma' \in W^{2,\infty}(0,\infty) \cap W^{2,1}(0,\infty). \tag{1.15}$$

Note that the assumption (1.13) means that $\hat{Q}$ is decreasing if $n > 2$ and increasing if $n \leqslant 2$ in the sense that when $t > t'$ and $n > 2$ then the projection of $\Omega_{t'}$ on the subspace $t = 0$ contains the projection of Ω_t on the same subspace and contrary in the case $n \leqslant 2$.

The above method was introduced by Dal Passo and Ughi [12] to study a certain class of parabolic equations in noncylindrical domains.

Remark. We only obtained the exponential decay of solution for our problem for the case $n > 2$. The main difficulty to prove the exponential decay for the case $n \leqslant 2$ is in Lemma 3.3, where it appear the terms

$$\begin{aligned} &-\int_{\Gamma_t} \frac{\gamma'}{\gamma}(\overline{\nu}\cdot x)\big(|u_t|^2 + |\nabla u|^2\big)\,d\Gamma_t - \int_{\Gamma_t} \frac{\gamma'}{\gamma}(\overline{\nu}\cdot x)\big(|v_t|^2 + |\nabla v|^2\big)\,d\Gamma_t \\ &\quad - \int_{\Gamma_t} \frac{\gamma'}{\gamma}(\overline{\nu}\cdot x)\int_0^t g_1(t-s)\big|\nabla u(t) - \nabla u(s)\big|^2\,ds\,d\Gamma_t \\ &\quad - \int_{\Gamma_t} \frac{\gamma'}{\gamma}(\overline{\nu}\cdot x)\int_0^t g_2(t-s)\big|\nabla v(t) - \nabla v(s)\big|^2\,ds\,d\Gamma_t \\ &\quad - \int_{\Gamma_t} \frac{\gamma'}{\gamma}(\overline{\nu}\cdot x) H(u-v)\,d\Gamma_t, \end{aligned}$$

since we worked directly in $\hat{Q}$. To control those terms we used the hypothesis (1.13). Therefore the case $n \leqslant 2$ is an important open problem.

To see the dissipative properties of the system we have to construct a suitable functional whose derivative is negative and is equivalent to the first order energy. This functional is obtained using the multiplicative technique following Komornik [6] or Rivera [10]. From the physics point of view, the problem (1.1)–(1.4) describes the transverse displacements of a stretched viscoelastic membrane fixed in a moving boundary device. The viscoelasticity property of the material is characterized by the memory terms

$$\int_0^t g_1(t-s)\Delta u(s)\,ds, \qquad \int_0^t g_2(t-s)\Delta v(s)\,ds.$$

In a fixed domain, the system of wave equations with coupled linear and nonlinear was studied by different authors. All of them consider essentially two types of dissipative mechanisms:

(a) the frictional dissipation, obtained by introducing a frictional damping that can act either on the boundary or in neighborhood of the boundary;
(b) the viscoelastic dissipation given by the memory effects as in [5,11,13,14].

The frictional damping is the simple dissipative mechanism when one is working either in the whole domain Ω. It was proved in [1–3,6] that the first-order energy decays exponentially to zero as time goes to infinity.

Finally, the memory effect produces a suitable dissipative mechanism which depends on the relaxation function (see [11,13,14]). They proved that the energy decays uniformly exponentially or algebraically with the same rate of decay as the relaxation function, that is, when the relaxation function decays exponentially, the corresponding solution also decays exponentially. On the other hand, when the relaxation function decays polynomially, the solution decays polynomially with the same rate.

But in a moving boundary setting, the axial tension exerted by the horizontal movement of the boundary yields nonlinear terms involving derivatives in the space variable. To control these nonlinearities, we add in the system a frictional damping, characterized by u_t and v_t. This term will play an important role in the dissipative nature of the problem.

The present paper is an extension to domains noncylindrical of the existent results in the literature.

We use the standard notations which can be found in Lions book [8,9]. In the sequel by C (sometimes $C_1, C_2, \ldots$) we denote various positive constants which do not depend on t or on the initial data.

This paper is organized as follows. In Section 2 we prove the existence, regularity and uniqueness of regular solutions. We use Galerkin approximation, Aubin–Lions theorem, energy method introduced by Lions [9] and some technical ideas to show existence regularity and uniqueness of regular solution for problem (1.1)–(1.4). Finally, in Section 3, we establish a result on the exponential decay of the regular solution to the problem (1.1)–(1.4). We use the technique of the multipliers introduced by Komornik [7], Lions [9] and Rivera [11] coupled with some technical lemmas and some technical ideas.

2. Existence and regularity

In this section we shall study the existence and regularity of solutions for the system (1.8)–(1.11). For this we assume that the kernel $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ is in $W^{2,1}(0,\infty)$, and satisfies

$$g_i > 0, \quad -g_i' > 0, \quad \gamma_1^{-2} - \int_0^\infty g_i(s)\gamma^{-2}(s)\,ds = \beta_i > 0, \quad i = 1, 2, \tag{2.1}$$

where

$$\gamma_1 = \sup_{0 \leqslant t < \infty} \gamma(t).$$

The above hypotheses (2.1) imply

$$\beta_i \leqslant \gamma(t)^{-2} - \int_0^t g_i(s)\gamma^{-2}(s)\,ds \leqslant \frac{1}{\gamma_0^2}.$$

We assume, also, that $h \in C^1(\mathbb{R})$ satisfies

$$h(s)s \geqslant 0, \quad \forall s \in \mathbb{R}.$$

Additionally, we suppose that h is superlinear, that is

$$h(s)s \geqslant (2+\delta)H(s), \quad H(z) = \int_0^z h(s)\,ds, \quad \forall s \in \mathbb{R},$$

for some $\delta > 0$ with the following growth conditions

$$\big|h(x) - h(y)\big| \leqslant C\big(1 + |x|^{\rho-1} + |y|^{\rho-1}\big)|x-y|, \quad \forall x, y \in \mathbb{R},$$

for some $C > 0$ and $\rho \geqslant 1$ such that $(n-2)\rho \leqslant n$. To simplify our analysis, we define the binary operator

$$g \,\square\, \frac{\nabla\phi(t)}{\gamma(t)} = \int_\Omega \int_0^t g(t-s)\gamma^{-2}(s)\big|\nabla\phi(t) - \nabla\phi(s)\big|^2\,ds\,dx.$$

With this notation we have the following statement.

Lemma 2.1. *For $\phi \in C^1(0, T : H^1(\Omega))$ and $g \in C^1((0,\infty) : \mathbb{R}^+)$ we have*

$$\int_\Omega \int_0^t g(t-s)\gamma^{-2}(s)\nabla\phi \cdot \nabla\phi_t\,ds\,dx = -\frac{1}{2}\frac{g(t)}{\gamma^2(0)}\int_\Omega |\nabla\phi|^2\,dx + \frac{1}{2}g' \,\square\, \frac{\nabla\phi}{\gamma} - \frac{1}{2}\frac{d}{dt}\left[g \,\square\, \frac{\nabla\phi}{\gamma} - \left(\int_0^t \frac{g(s)}{\gamma^2(s)}\,ds\right)\int_\Omega |\nabla\phi|^2\,dx\right].$$

The proof of this lemma follows by differentiating the term $g \,\square\, \frac{\nabla\phi(t)}{\gamma(t)}$. The well-posedness of system (1.8)–(1.11) is given by the following theorem.

Theorem 2.1. *Let us take $(\varphi_0, \psi_0) \in (H_0^1(\Omega) \cap H^2(\Omega))^2$, $(\varphi_1, \psi_1) \in (H_0^1(\Omega))^2$ and let us suppose that assumptions* (1.13)–(1.15) *and* (2.1) *hold. Then there exists a unique solution (φ, ψ) of the problem* (1.8)–(1.11) *satisfying*

$$\varphi, \psi \in L^\infty\big(0, \infty : H_0^1(\Omega) \cap H^2(\Omega)\big),$$
$$\varphi_t, \psi_t \in L^\infty\big(0, \infty : H_0^1(\Omega)\big),$$
$$\varphi_{tt}, \psi_{tt} \in L^\infty\big(0, \infty : L^2(\Omega)\big).$$

Proof. Let us denote by B the operator

$$Bw = -\Delta w, \quad D(B) = H_0^1(\Omega) \cap H^2(\Omega).$$

It is well know that B is a positive self-adjoint operator in the Hilbert space $L^2(\Omega)$ for which there exist sequences $\{w_m\}_{m\in\mathbb{N}}$ and $\{\lambda_m\}_{m\in\mathbb{N}}$ of eigenfunctions and eigenvalues of B such that the set of linear combinations of $\{w_m\}_{m\in\mathbb{N}}$ is dense in $D(B)$ and $\lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_m \to \infty$ as $m \to \infty$. Let us denote by

$$\varphi_0^m = \sum_{j=1}^{m} (\varphi_0, w_j) w_j, \qquad \psi_0^m = \sum_{j=1}^{m} (\psi_0, w_j) w_j,$$
$$\varphi_1^m = \sum_{j=1}^{m} (\varphi_1, w_j) w_j, \qquad \psi_1^m = \sum_{j=1}^{m} (\psi_1, w_j) w_j.$$

Note that for any $\{(\varphi_0, \psi_0), (\varphi_1, \psi_1)\} \in (D(B))^2 \times (H_0^1(\Omega))^2$, we have $(\varphi_0^m, \psi_0^m) \to (\varphi_0, \psi_0)$ strong in $(D(B))^2$ and $(\varphi_1^m, \psi_1^m) \to (\varphi_1, \psi_1)$ strong in $(H_0^1(\Omega))$.

Let us denote by V_m the space generated by $w_1, \dots, w_m$. Standard results on ordinary differential equations imply the existence of a local solution (φ^m, ψ^m) of the form

$$\big(\varphi^m(t), \psi^m(t)\big) = \sum_{j=1}^{m} \big(h_{jm}(t), f_{jm}(t)\big) w_j,$$

to the system

$$\begin{aligned}
&\int_\Omega \varphi_{tt}^m w_j\,dy + \alpha \int_\Omega \varphi_t^m w_j\,dy - \gamma^{-2} \int_\Omega \Delta\varphi^m w_j\,dy + \int_\Omega h\big(\varphi^m - \psi^m\big) w_j\,dy \\
&\quad + \int_\Omega \int_0^t g_1(t-s)\gamma^{-2}(s) \nabla\varphi^m(s) \cdot \nabla w_j\,ds\,dy + \int_\Omega A(t)\varphi^m w_j\,dy \\
&\quad + \int_\Omega a_1 \cdot \nabla\varphi_t^m w_j\,dy + \int_\Omega a_2 \cdot \nabla\varphi^m w_j\,dy = 0 \quad (j = 1, \dots, m),
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&\int_\Omega \psi_{tt}^m w_j\,dy + \alpha \int_\Omega \psi_t^m w_j\,dy - \gamma^{-2} \int_\Omega \Delta\psi^m w_j\,dy - \int_\Omega h\big(\varphi^m - \psi^m\big) w_j\,dy \\
&\quad + \int_\Omega \int_0^t g_2(t-s)\gamma^{-2}(s) \nabla\psi^m(s) \cdot \nabla w_j\,ds\,dy + \int_\Omega A(t)\psi^m w_j\,dy \\
&\quad + \int_\Omega a_1 \cdot \nabla\psi_t^m w_j\,dy + \int_\Omega a_2 \cdot \nabla\psi^m w_j\,dy = 0 \quad (j = 1, \dots, m),
\end{aligned} \tag{2.3}$$

$$\big(\varphi^m(x,0), \psi^m(x,0)\big) = \big(\varphi_0^m, \psi_0^m\big), \qquad \big(\varphi_t^m(x,0), \psi_t^m(x,0)\big) = \big(\varphi_1^m, \psi_1^m\big). \tag{2.4}$$

See [4] for details.

The extension of these solutions to the whole interval $[0, T]$, $0 < T < \infty$, is a consequence of the first estimate which we are going to prove below.

A priori estimate I: Multiplying Eqs. (2.2)–(2.3) by $h'_{jm}(t)$ and f'_{jm}, respectively, summing up the product result and using Lemma 2.1 we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\pounds_1^m\big(t, \varphi^m, \psi^m\big) + \alpha\big(\|\varphi_t^m\|_{L^2(\Omega)}^2 + \|\psi_t^m\|_{L^2(\Omega)}^2\big) + \int_\Omega A(t)\varphi^m \varphi_t^m\,dy \\
&\quad + \int_\Omega A(t)\psi^m \psi_t^m\,dy + \int_\Omega a_1 \cdot \nabla\varphi_t^m \varphi_t^m\,dy + \int_\Omega a_1 \cdot \nabla\psi_t^m \psi_t^m\,dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega a_2\cdot\nabla\varphi^m\varphi_t^m\,dy+\int_\Omega a_2\cdot\nabla\psi^m\psi_t^m\,dy\\
&=-\frac{1}{2}\frac{g_1(t)}{\gamma^2(0)}\|\nabla\varphi^m\|^2_{L^2(\Omega)}\\
&\quad-\frac{1}{2}\frac{g_2(t)}{\gamma^2(0)}\|\nabla\psi^m\|^2_{L^2(\Omega)}+\frac{1}{2}g_1'\,\Box\,\frac{\nabla\varphi^m}{\gamma}+\frac{1}{2}g_2'\,\Box\,\frac{\nabla\psi^m}{\gamma}\\
&\quad-\frac{\gamma'}{\gamma^3}\|\nabla\varphi^m\|^2_{L^2(\Omega)}-\frac{\gamma'}{\gamma^3}\|\nabla\psi^m\|^2_{L^2(\Omega)},
\end{aligned}
$$

where

$$
\begin{aligned}
&\pounds_1^m\big(t,\varphi^m,\psi^m\big)\\
&=\|\varphi_t^m\|^2_{L^2(\Omega)}+\left(\frac{1}{\gamma^2(t)}-\int_0^t g_1(s)\gamma^{-2}(s)\,ds\right)\|\nabla\varphi^m\|^2_{L^2(\Omega)}\\
&\quad+g_1\,\Box\,\frac{\nabla\varphi^m}{\gamma}+\|\psi_t^m\|^2_{L^2(\Omega)}+\left(\frac{1}{\gamma^2(t)}-\int_0^t g_2(s)\gamma^{-2}(s)\,ds\right)\|\nabla\psi^m\|^2_{L^2(\Omega)}\\
&\quad+g_2\,\Box\,\frac{\nabla\psi^m}{\gamma}+\int_\Omega H(\varphi-\psi)\,dy.
\end{aligned}
$$

Taking into account (1.11), (1.13) and (2.1) we obtain

$$
\frac{1}{2}\frac{d}{dt}\pounds_1^m\big(t,\varphi^m,\psi^m\big)+\alpha\big(\|\varphi_t^m\|^2_{L^2(\Omega)}+\|\psi_t^m\|^2_{L^2(\Omega)}\big)\leqslant C\big(|\gamma'|+|\gamma''|\big)\pounds_1^m(t). \tag{2.5}
$$

Integrating the inequality (2.5), using Gronwall's lemma and taking into account (1.13) we get

$$
\pounds_1^m\big(t,\varphi^m,\psi^m\big)+\int_0^t\big(\|\varphi_s^m(s)\|^2_{L^2(\Omega)}+\|\psi_s^m(s)\|^2_{L^2(\Omega)}\big)\,ds\leqslant C,\quad \forall m\in\mathbb{N},\ \forall t\in[0,T]. \tag{2.6}
$$

A priori estimate II: From Eqs. (2.2) and (2.3) we get

$$
\|\varphi_{tt}^m(0)\|^2_{L^2(\Omega)}+\|\psi_{tt}^m(0)\|^2_{L^2(\Omega)}\leqslant C,\quad \forall m\in\mathbb{N}. \tag{2.7}
$$

Using the hypotheses about g_i and γ, and differentiating Eqs. (2.2) and (2.3) with respect to the time, we obtain

$$
\begin{aligned}
&\int_\Omega\varphi_{ttt}^m w_j\,dy+\alpha\int_\Omega\varphi_{tt}^m w_j\,dy-\gamma^{-2}\int_\Omega\Delta\varphi_t^m w_j\,dy\\
&\quad+2\frac{\gamma'}{\gamma^3}\int_\Omega\Delta\varphi^m w_j\,dy-\frac{g_1(t)}{\gamma^2(0)}\int_\Omega\Delta\varphi_0^m w_j\,dy\\
&\quad+\int_\Omega\int_0^t g_1'(t-s)\gamma^{-2}(s)\nabla\varphi^m(s)\,ds\cdot\nabla w_j\,dy
\end{aligned}
$$

$$+\int_\Omega \frac{d}{dt}\big(A(t)\varphi^m\big)w_j\,dy+\int_\Omega \frac{d}{dt}\big(a_1\cdot\nabla\varphi_t^m\big)w_j\,dy$$

$$+\int_\Omega \frac{d}{dt}\big(a_2\cdot\nabla\varphi^m\big)w_j\,dy+\int_\Omega h'(\varphi-\psi)(\varphi_t-\psi_t)w_j\,dy=0, \tag{2.8}$$

$$\int_\Omega \psi_{ttt}^m w_j\,dy+\alpha\int_\Omega \psi_{tt}^m w_j\,dy-\gamma^{-2}\int_\Omega \Delta\psi_t^m w_j\,dy$$

$$+2\frac{\gamma'}{\gamma^3}\int_\Omega \Delta\psi^m w_j\,dy-\frac{g_2(t)}{\gamma^2(0)}\int_\Omega \Delta\psi_0^m w_j\,dy$$

$$+\int_\Omega\int_0^t g_2'(t-s)\gamma^{-2}(s)\nabla\psi^m(s)\,ds\cdot\nabla w_j\,dy$$

$$+\int_\Omega \frac{d}{dt}\big(A(t)\psi^m\big)w_j\,dy+\int_\Omega \frac{d}{dt}\big(a_1\cdot\nabla\psi_t^m\big)w_j\,dy$$

$$+\int_\Omega \frac{d}{dt}\big(a_2\cdot\nabla\psi^m\big)w_j\,dy-\int_\Omega h'(\varphi-\psi)(\varphi_t-\psi_t)w_j\,dy=0. \tag{2.9}$$

Multiplying (2.8) by $h''_{jm}(t)$ and (2.9) by $f''_{jm}(t)$, summing up the product result we obtain

$$\int_\Omega \varphi_{ttt}^m\varphi_{tt}^m\,dy+\alpha\int_\Omega \varphi_{tt}^m\varphi_{tt}^m\,dy-\gamma^{-2}\int_\Omega \Delta\varphi_t^m\varphi_{tt}^m\,dy$$

$$+2\frac{\gamma'}{\gamma^3}\int_\Omega \Delta\varphi^m\varphi_{tt}^m\,dy-\frac{g_1(t)}{\gamma^2(0)}\int_\Omega \Delta\varphi_0^m\varphi_{tt}^m\,dy$$

$$+\int_\Omega\int_0^t g_1'(t-s)\gamma^{-2}(s)\nabla\varphi^m(s)\,ds\cdot\nabla\varphi_{tt}^m\,dy$$

$$+\int_\Omega \frac{d}{dt}\big(A(t)\varphi^m\big)\varphi_{tt}^m\,dy+\int_\Omega \frac{d}{dt}\big(a_1\cdot\nabla\varphi_t^m\big)\varphi_{tt}^m\,dy$$

$$+\int_\Omega \frac{d}{dt}\big(a_2\cdot\nabla\varphi^m\big)\varphi_{tt}^m\,dy+\int_\Omega h'(\varphi-\psi)(\varphi_t-\psi_t)\varphi_{tt}^m\,dy$$

$$+\int_\Omega \psi_{ttt}^m w_j\,dy+\alpha\int_\Omega \psi_{tt}^m\psi_{tt}^m\,dy-\gamma^{-2}\int_\Omega \Delta\psi_t^m\psi_{tt}^m\,dy$$

$$+2\frac{\gamma'}{\gamma^3}\int_\Omega \Delta\psi^m\psi_{tt}^m\,dy-\frac{g_2(t)}{\gamma^2(0)}\int_\Omega \Delta\psi_0^m\psi_{tt}^m\,dy$$

$$+\int_\Omega\int_0^t g_2'(t-s)\gamma^{-2}(s)\nabla\psi^m(s)\,ds\cdot\nabla\psi_{tt}^m\,dy$$

$$+\int_\Omega \frac{d}{dt}\big(A(t)\psi^m\big)\psi_{tt}^m\,dy+\int_\Omega \frac{d}{dt}\big(a_1\cdot\nabla\psi_t^m\big)\psi_{tt}^m\,dy$$
$$+\int_\Omega \frac{d}{dt}\big(a_2\cdot\nabla\psi^m\big)\psi_{tt}^m\,dy-\int_\Omega h'(\varphi-\psi)(\varphi_t-\psi_t)\psi_{tt}^m\,dy=0. \tag{2.10}$$

Let us take $p_n=\frac{2n}{n-2}$. From the growth condition of the function h and of the Sobolev imbedding we have

$$\begin{aligned}
&\int_\Omega h'\big(\varphi^m-\psi^m\big)\varphi_t^m\varphi_{tt}^m\,dy\\
&\quad\leqslant C\int_\Omega\big(1+2\big|\varphi^m-\psi^m\big|^{\rho-1}\big)\big|\varphi_t^m\big|\big|\varphi_{tt}^m\big|\,dy\\
&\quad\leqslant C\bigg[\int_\Omega\big(1+2\big|\varphi^m-\psi^m\big|^{\rho-1}\big)^{\rho-1}\,dy\bigg]^{\frac{1}{n}}\bigg[\int_\Omega\big|\varphi_t^m\big|^{p_n}\,dy\bigg]^{\frac{1}{p_n}}\bigg[\int_\Omega\big|\varphi_{tt}^m\big|^2\,dy\bigg]^{\frac12}\\
&\quad\leqslant C\bigg[\int_\Omega\big(1+\big|\nabla\varphi^m-\nabla\psi^m\big|^2\big)\,dy\bigg]^{\frac{\rho-1}{2}}\bigg[\int_\Omega\big|\nabla\varphi_t^m\big|^2\,dy\bigg]^{\frac12}\bigg[\int_\Omega\big|\varphi_{tt}^m\big|^2\,dy\bigg]^{\frac12}.
\end{aligned}$$

Taking into account the first estimate (2.6) and using the elementary inequality we conclude that

$$\int_\Omega h'\big(\varphi^m-\psi^m\big)\varphi_t^m\varphi_{tt}^m\,dy\leqslant C\bigg\{\int_\Omega\big|\nabla\varphi_t^m\big|^2\,dy+\int_\Omega|\varphi_{tt}|^2\,dy\bigg\}. \tag{2.11}$$

Similarly we get

$$-\int_\Omega h'\big(\varphi^m-\psi^m\big)\psi_t^m\varphi_{tt}^m\,dy\leqslant C\bigg\{\int_\Omega\big|\nabla\psi_t^m\big|^2\,dy+\int_\Omega|\varphi_{tt}|^2\,dy\bigg\}, \tag{2.12}$$

$$\int_\Omega h'\big(\varphi^m-\psi^m\big)\varphi_t^m\psi_{tt}^m\,dy\leqslant C\bigg\{\int_\Omega\big|\nabla\varphi_t^m\big|^2\,dy+\int_\Omega|\psi_{tt}|^2\,dy\bigg\}, \tag{2.13}$$

$$-\int_\Omega h'\big(\varphi^m-\psi^m\big)\psi_t^m\psi_{tt}^m\,dy\leqslant C\bigg\{\int_\Omega\big|\nabla\psi_t^m\big|^2\,dy+\int_\Omega|\psi_{tt}|^2\,dy\bigg\}. \tag{2.14}$$

Substituting the inequalities (2.11)–(2.14) into (2.10) and using similar arguments as (2.6) we obtain

$$\begin{aligned}
&\pounds_2^m\big(t,\varphi_t^m,\psi_t^m\big)+\int_0^t\big(\big\|\varphi_{ss}^m(s)\big\|_{L^2(\Omega)}^2+\big\|\psi_{ss}^m(s)\big\|_{L^2(\Omega)}^2\big)\,ds\leqslant C,\\
&\forall t\in[0,T],\ \forall m\in\mathbb{N},
\end{aligned} \tag{2.15}$$

where

$$\pounds_2^m\big(t,\varphi^m,\psi^m\big)=\big\|\varphi_t^m\big\|_{L^2(\Omega)}^2+\left(\frac{1}{\gamma^2(t)}-\int_0^t g_1(s)\gamma^{-2}(s)\,ds\right)\big\|\nabla\varphi^m\big\|_{L^2(\Omega)}^2$$

$$+ g_1 \Box \frac{\nabla \varphi^m}{\gamma} \|\psi_t^m\|^2_{L^2(\Omega)} + \left(\frac{1}{\gamma^2(t)} - \int_0^t g_2(s)\gamma^{-2}(s)\,ds \right) \|\nabla \psi^m\|^2_{L^2(\Omega)}$$
$$+ g_2 \Box \frac{\nabla \psi^m}{\gamma}.$$

The estimates (2.6) and (2.15) permit to obtain a subsequence $(\varphi^{m_k}, \psi^{m_k})$ of (φ^m, ψ^m), which will be all denoted by (φ^m, ψ^m) and the functions $\varphi, \psi : \Omega \times]0, \infty[\to \mathbb{R}$ satisfying:

$$\big(\varphi^m, \psi^m\big) \to (\varphi, \psi) \quad \text{weak star in } L^\infty\big(0, \infty : H_0^1(\Omega)\big),$$
$$\big(\varphi_t^m, \psi_t^m\big) \to (\varphi_t, \psi_t) \quad \text{weak star in } L^\infty\big(0, \infty : H_0^1(\Omega)\big),$$
$$\big(\varphi_{tt}^m, \psi_{tt}^m\big) \to (\varphi_{tt}, \psi_{tt}) \quad \text{weak star in } L^\infty\big(0, \infty : L^2(\Omega)\big).$$

Letting $m \to \infty$ in Eqs. (2.2)–(2.3) and using the above estimates we conclude that (φ, ψ) satisfies (1.8)–(1.9) in the sense of $L^\infty(0, \infty : L^2(\Omega))$. Therefore, using the elliptic regularity, we have that

$$\varphi, \psi \in L^\infty\big(0, \infty : H_0^1(\Omega) \cap H^2(\Omega)\big).$$

Uniqueness. Let us suppose we have two solutions (φ, ψ) and $(\hat{\varphi}, \hat{\psi})$ in the conditions of Theorem 2.1. Then $(\phi, \theta) = (\varphi - \hat{\varphi}, \psi - \hat{\psi})$ satisfies the same conditions and $(\phi(0), \theta(0)) = (0, 0)$, $(\phi_t(0), \theta_t(0)) = (0, 0)$. Let us prove that $(\phi, \theta) = (0, 0)$ on $\Omega \times [0, \infty[$.

Multiplying Eqs. (1.8) and (1.9) by ϕ_t and θ_t, respectively, summing up the product result and using Lemma 2.1, the growth condition of the function h and the Sobolev imbedding we get

$$\frac{1}{2}\frac{d}{dt}\pounds_1(t, \phi, \theta) + \alpha\big(\|\phi_t\|^2_{L^2(\Omega)} + \|\theta_t\|^2_{L^2(\Omega)}\big) \leqslant C\big(|\gamma'| + |\gamma''|\big)\pounds_1(t),$$

where

$$\pounds_1(t, \phi, \theta) = \|\phi_t\|^2_{L^2(\Omega)} + \left(\frac{1}{\gamma^2(t)} - \int_0^t g_1(s)\gamma^{-2}(s)\,ds \right) \|\nabla \phi\|^2_{L^2(\Omega)}$$
$$+ g_1 \Box \frac{\nabla \phi}{\gamma} + \|\theta_t\|^2_{L^2(\Omega)} + \left(\frac{1}{\gamma^2(t)} - \int_0^t g_2(s)\gamma^{-2}(s)\,ds \right) \|\nabla \theta\|^2_{L^2(\Omega)}$$
$$+ g_2 \Box \frac{\nabla \theta}{\gamma}.$$

Integrating with respect to the time the above inequality and applying Gronwall's inequality we conclude that $(\phi, \theta) = (0, 0)$ on $\Omega \times [0, \infty[$. □

To show the existence in noncylindrical domains, we return to our original problem in the noncylindrical domains by using the change variable given in (1.5) by $(y, t) = \tau(x, t)$, $(x, t) \in \hat{Q}$. Let (φ, ψ) be the solution obtained from Theorem 2.1 and (u, v) defined by (1.7), then (u, v) belongs to the class

$$u, v \in L^\infty\big(0, \infty : H_0^1(\Omega_t)\big), \tag{2.16}$$
$$u_t, v_t \in L^\infty\big(0, \infty : H_0^1(\Omega_t)\big), \tag{2.17}$$
$$u_{tt}, v_{tt} \in L^\infty\big(0, \infty : L^2(\Omega_t)\big). \tag{2.18}$$

Denoting by

$$u(x,t)=\varphi(y,t)=(\varphi\circ\tau)(x,t), \qquad v(x,t)=\psi(y,t)=(\psi\circ\tau)(x,t)$$

then from (1.6) it is easy to see that (u,v) satisfies Eqs. (1.1)–(1.2) in the sense of $L^\infty(0,\infty: L^2(\Omega_t))$. Let (u_1,v_1), (u_2,v_2) be two solutions to (1.1)–(1.2), and (φ_1,ψ_1), (φ_2,ψ_2) be the functions obtained through the diffeomorphism τ given by (1.5). Then (φ_1,ψ_1), (φ_2,ψ_2) are the solutions to (1.8)–(1.9). By the uniqueness result Theorem 2.1, we have $(\varphi_1,\psi_1)=(\varphi_2,\psi_2)$, so $(u_1,v_1)=(u_2,v_2)$. Therefore, we have the following result.

Theorem 2.2. *Let us take* $(u_0,v_0)\in(H_0^1(\Omega_0)\cap H^2(\Omega_0))^2$, $(u_1,v_1)\in(H_0^1(\Omega_0))^2$ *and let us suppose that assumptions* (1.11)–(1.13) *and* (2.1) *hold. Then there exists a unique solution* (u,v) *of the problem* (1.1)–(1.4) *satisfying* (2.16)–(2.18) *and Eqs.* (1.1)–(1.2) *in* $L^\infty(0,\infty:L^2(\Omega_t))$.

3. Exponential decay

In this section we show that the solution of system (1.1)–(1.4) decays exponentially. To this end we will assume that the memory g_i satisfies:

$$g_i'(t)\leqslant -C_1 g_i(t), \tag{3.1}$$

$$\left(1-\int_0^\infty g(s)\,ds\right)=\eta_i, \quad \forall i=1,2 \tag{3.2}$$

for all $t\geqslant 0$, with positive constant C_1. Additionally, we assume that the function $\gamma(\cdot)$ satisfies the conditions

$$\gamma'\leqslant 0, \quad t\geqslant 0, \quad n>2, \tag{3.3}$$

$$0<\max_{0\leqslant t<\infty}\left|\gamma'(t)\right|\leqslant\frac{1}{d}, \tag{3.4}$$

where $d=\operatorname{diam}(\Omega)$. Condition (3.4) implies that our domain is "time like" in the sense that

$$|\underline{\nu}|<|\overline{\nu}|,$$

where $\underline{\nu}$ and $\overline{\nu}$ denote the t-component and x-component of the outer unit normal of $\hat{\Sigma}$. To facilitate our calculations we introduce the following notation:

$$(g\,\Box\,\nabla u)(t)=\int_{\Omega_t}\int_0^t g(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 ds\,dx.$$

Due to geometry of our noncylindrical domain, the below lemma is quite important in the proof of Lemma 3.2 and of the exponential decay.

Lemma 3.1. *Let $F(\cdot,\cdot)$ be the smooth function defined in $\Omega_t\times[0,\infty[$ ($t\in[0,\infty[$). Then*

$$\frac{d}{dt}\int_{\Omega_t}F(x,t)\,dx=\int_{\Omega_t}\frac{d}{dt}F(x,t)\,dx+\frac{\gamma'}{\gamma}\int_{\Gamma_t}F(x,t)(x\cdot\overline{\nu})\,d\Gamma_t, \tag{3.5}$$

where $\overline{\nu}$ is the x-component of the unit normal exterior ν.

Proof. We have by a change of variable $x=\gamma(t)y$, $y\in\Omega$,

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t}F(x,t)\,dx&=\frac{d}{dt}\int_{\Omega}F\big(\gamma(t)y,t\big)\gamma^n(t)\,dy\\&=\int_{\Omega}\left(\frac{\partial F}{\partial t}\right)\gamma^n(t)\,dy+\sum_{i=1}^{n}\int_{\Omega}\frac{\gamma'}{\gamma}x_i\left(\frac{\partial F}{\partial t}\right)\gamma^n(t)\,dy\\&\quad+n\int_{\Omega}\gamma'(t)\gamma^{n-1}(t)F\big(\gamma(t)y,t\big)\,dy.\end{aligned}$$

If we return at the variable x, we get

$$\frac{d}{dt}\int_{\Omega_t}F(x,t)\,dx=\int_{\Omega_t}\frac{\partial F}{\partial t}\,dx+\frac{\gamma'}{\gamma}\int_{\Omega_t}x\cdot\nabla F(x,t)\,dx+n\frac{\gamma'}{\gamma}\int_{\Omega_t}F(x,t)\,dx.$$

Integrating by parts the last equality we obtain the formula (3.5). $\square$

Lemma 3.2. *For any function $g\in C^1(\mathbb{R}^+)$ and $u\in C^1((0,T):H^2(\Omega_t))$ we have that*

$$\begin{aligned}\int_{\Omega_t}\int_0^t g(t-s)\nabla u(s)\,ds\cdot\nabla u_t\,dx&=-\frac{1}{2}g(t)\int_{\Omega_t}\big|\nabla u(t)\big|^2\,dx+\frac{1}{2}g'\,\Box\,\nabla u\\&\quad-\frac{1}{2}\frac{d}{dt}\left[g\,\Box\,\nabla u-\left(\int_0^t g(s)\,ds\right)\int_{\Omega_t}|\nabla u|^2\right]\\&\quad+\frac{\gamma'}{2\gamma}\int_{\Gamma_t}\int_0^t g(t-s)\big|\nabla u(t)-\nabla u(s)\big|^2(\overline{\nu}\cdot x)\,d\Gamma_t.\end{aligned}$$

Proof. Differentiating the term $g\,\Box\,\nabla u$ and applying Lemma 3.1 we obtain

$$\begin{aligned}\frac{d}{dt}g\,\Box\,\nabla u&=\int_{\Omega_t}\frac{d}{dt}\int_0^t g(t-s)\big|\nabla u(t)-\nabla u(s)\big|^2\,ds\,dx\\&\quad\times\frac{\gamma'}{\gamma}\int_{\Gamma_t}\int_0^t g(t-s)\big|\nabla u(t)-\nabla u(s)\big|^2\,ds\,d\Gamma_t.\end{aligned}$$

Noting that

$$\begin{aligned}&\int_{\Omega_t}\frac{d}{dt}\int_0^t g(t-s)\big|\nabla u(t)-\nabla u(s)\big|^2\,ds\,dx\\&\quad=-2\int_{\Omega_t}\int_0^t g(t-s)\nabla u(s)\cdot\nabla u_t\,ds\,dx\end{aligned}$$

$$- g(t) \int_{\Omega_t} \left|\nabla u(t)\right|^2 dx + g' \,\square\, \nabla u + \frac{d}{dt}\left(\int_0^t g(s)ds\right) \int_{\Omega_t} |\nabla u|^2 \, dx$$

follows the conclusion of the lemma. □

Let us introduce the functional

$$\begin{aligned} E(t) &= \|u_t\|^2_{L^2(\Omega_t)} + \left(1 - \int_0^t g_1(s)\, ds\right) \|\nabla u\|^2_{L^2(\Omega_t)} + g_1 \,\square\, \nabla u \\ &\quad + \|v_t\|^2_{L^2(\Omega_t)} + \left(1 - \int_0^t g_2(s)\, ds\right) \|\nabla v\|^2_{L^2(\Omega_t)} + g_2 \,\square\, \nabla v \\ &\quad + \int_{\Omega_t} H(u - v)\, dx. \end{aligned}$$

We observe that $E(t) > 0$ since hypothesis (3.2) is satisfied.

The following lemma shows the dissipative property of the energy of system (1.1)–(1.4). For this hypothesis (1.13) it is a crucial point.

Lemma 3.3. *Let us take* $(u_0, v_0) \in (H_0^1(\Omega_0) \cap H^2(\Omega_0))^2$, $(u_1, v_1) \in (H_0^1(\Omega_0))^2$ *and let us suppose that assumptions* (1.11)–(1.13) *and* (2.1) *hold. Then any regular of system* (1.1)–(1.4) *satisfies*

$$\begin{aligned} &\frac{d}{dt} E(t) + 2\alpha\left(\|u_t\|^2_{L^2(\Omega_t)} + \|v_t\|^2_{L^2(\Omega_t)}\right) - \int_{\Gamma_t} \frac{\gamma'}{\gamma} (\bar{\nu} \cdot x)\left(|u_t|^2 + |\nabla u|^2\right) d\Gamma_t \\ &\quad - \int_{\Gamma_t} \frac{\gamma'}{\gamma} (\bar{\nu} \cdot x)\left(|v_t|^2 + |\nabla v|^2\right) d\Gamma_t \\ &\quad - \int_{\Gamma_t} \frac{\gamma'}{\gamma} (\bar{\nu} \cdot x) \int_0^t g_1(t-s)\left|\nabla u(t) - \nabla u(s)\right|^2 ds\, d\Gamma_t \\ &\quad - \int_{\Gamma_t} \frac{\gamma'}{\gamma} (\bar{\nu} \cdot x) \int_0^t g_2(t-s)\left|\nabla v(t) - \nabla v(s)\right|^2 ds\, d\Gamma_t - \int_{\Gamma_t} \frac{\gamma'}{\gamma} (\bar{\nu} \cdot x) H(u - v)\, d\Gamma_t \\ &= - \int_{\Omega_t} g_1(t) |\nabla u|^2 \, dx + g_1' \,\square\, \nabla u - \int_{\Omega_t} g_2(t) |\nabla v|^2 \, dx + g_2' \,\square\, \nabla v. \end{aligned}$$

Proof. Multiplying Eqs. (1.1) by u_t and (1.2) by v_t, performing an integration by parts over Ω_t and using Lemma 3.1 we get

$$\frac{1}{2} \frac{d}{dt} \|u_t\|^2_{L^2(\Omega_t)} + \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2_{L^2(\Omega_t)} + \alpha \|u_t\|^2_{L^2(\Omega_t)}$$

$$
\begin{aligned}
&- \int_{\Omega_t} \int_0^t g_1(t-s) \nabla u(s) \cdot \nabla u_t \, ds \, dx \\
&- \int_{\Gamma_t} \frac{\gamma'}{2\gamma} (\overline{\nu} \cdot x) \big(|u_t|^2 + |\nabla u|^2 \big) \, d\Gamma_t, \\
&\frac{1}{2} \frac{d}{dt} \|v_t\|^2_{L^2(\Omega_t)} + \frac{1}{2} \frac{d}{dt} \|\nabla v\|^2_{L^2(\Omega_t)} + \alpha \|v_t\|^2_{L^2(\Omega_t)} \\
&- \int_{\Omega_t} \int_0^t g_2(t-s) \nabla v(s) \cdot \nabla v_t \, ds \, dx \\
&- \int_{\Gamma_t} \frac{\gamma'}{2\gamma} (\overline{\nu} \cdot x) \big(|v_t|^2 + |\nabla v|^2 \big) \, d\Gamma_t + \int_{\Omega_t} \frac{d}{dt} H(u-v) \, dx.
\end{aligned}
$$

Taking into account Lemmas 3.1 and 3.2 we obtain the conclusion of the lemma. □

Let us consider the following functional:

$$
\psi(t) = 2 \int_{\Omega_t} (u_t u + v_t v) \, dx + \alpha \big(\|u\|^2_{L^2(\Omega_t)} + \|v\|^2_{L^2(\Omega_t)} \big).
$$

Lemma 3.4. *Let us take* $(u_0, v_0) \in (H_0^1(\Omega_0) \cap H^2(\Omega_0))^2$, $(u_1, v_1) \in (H_0^1(\Omega_0))^2$ *and let us suppose that assumptions* (1.11)–(1.13) *and* (2.1) *hold. Then any regular of system* (1.1)–(1.4) *satisfies*

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \psi(t) \leqslant{} & \|u_t\|^2_{L^2(\Omega_t)} - \|\nabla u\|^2_{L^2(\Omega_t)} + \left(\int_0^t g_1(s) \, ds \right) \|\nabla u\|^2_{L^2(\Omega_t)} \\
& + \|\nabla u\|_{L^2(\Omega_t)} \left(\int_0^t g_1(s) \, ds \right)^{\frac{1}{2}} (g_1 \Box \nabla u)^{\frac{1}{2}} \\
& + \|v_t\|^2_{L^2(\Omega_t)} - \|\nabla v\|^2_{L^2(\Omega_t)} + \left(\int_0^t g_2(s) \, ds \right) \|\nabla v\|^2_{L^2(\Omega_t)} \\
& + \|\nabla v\|_{L^2(\Omega_t)} \left(\int_0^t g_2(s) \, ds \right)^{\frac{1}{2}} (g_2 \Box \nabla v)^{\frac{1}{2}} \\
& - (2+\delta) \int_{\Omega_t} H(u-v) \, dx.
\end{aligned}
$$

Proof. Multiplying Eqs. (1.1) by u and (1.2) by v, integrating over Ω_t we obtain

$$\frac{1}{2}\frac{d}{dt}\psi(t)=\|u_t\|^2_{L^2(\Omega_t)}-\|\nabla u\|^2_{L^2(\Omega_t)}+\int_{\Omega_t}\int_0^t g_1(t-s)\nabla u(s)\cdot\nabla u\,ds\,dx$$
$$\times\|v_t\|^2_{L^2(\Omega_t)}-\|\nabla v\|^2_{L^2(\Omega_t)}+\int_{\Omega_t}\int_0^t g_2(t-s)\nabla v(s)\cdot\nabla u\,ds\,dx$$
$$-\int_{\Omega_t}(u-v)h(u-v)\,dx.$$

Noting that

$$\int_{\Omega_t}\int_0^t g_1(t-s)\nabla u(s)\cdot\nabla u\,ds\,dx=\int_{\Omega_t}\int_0^t g_1(t-s)\big(\nabla u(s)-\nabla u(t)\big)\cdot\nabla u\,ds\,dx+\int_{\Omega_t}\left(\int_0^t g_1(s)\,ds\right)|\nabla u|^2\,dx,$$
$$\int_{\Omega_t}\int_0^t g_2(t-s)\nabla v(s)\cdot\nabla v\,ds\,dx=\int_{\Omega_t}\int_0^t g_2(t-s)\big(\nabla v(s)-\nabla v(t)\big)\cdot\nabla v\,ds\,dx+\int_{\Omega_t}\left(\int_0^t g_2(s)\,ds\right)|\nabla v|^2\,dx$$

and taking into account that

$$\left|\int_{\Omega_t}\int_0^t g_1(t-s)\big(\nabla u(s)-\nabla u(t)\big)\cdot\nabla u\,dx\right|\leqslant\|\nabla u\|_{L^2(\Omega_t)}\left(\int_0^t g_1(s)\,ds\right)^{\frac{1}{2}}(g_1\,\square\,\nabla u)^{\frac{1}{2}},$$
$$\left|\int_{\Omega_t}\int_0^t g_2(t-s)\big(\nabla v(s)-\nabla v(t)\big)\cdot\nabla v\,dx\right|\leqslant\|\nabla v\|_{L^2(\Omega_t)}\left(\int_0^t g_2(s)\,ds\right)^{\frac{1}{2}}(g_2\,\square\,\nabla v)^{\frac{1}{2}},$$
$$-\int_{\Omega_t}(u-v)h(u-v)\,dx\leqslant-(2+\delta)\int_{\Omega_t}H(u-v)\,dx$$

follows the conclusion of the lemma. □

Let us introduce the functional

$$\mathcal{L}(t)=NE(t)+\psi(t),\tag{3.6}$$

with $N>0$. It is not difficult to see that $\mathcal{L}(t)$ verifies

$$k_0E(t)\leqslant\mathcal{L}(t)\leqslant k_1E(t),\tag{3.7}$$

for k_0 and k_1 positive constants. Now we are in a position to show the main result of this paper.

Theorem 3.1. *Let us take* $(u_0, v_0) \in (H_0^1(\Omega_0) \cap H^2(\Omega_0))^2$*,* $(u_1, v_1) \in (H_0^1(\Omega_0))^2$ *and let us suppose that assumptions* (1.12)*,* (1.13)*,* (2.1)*,* (3.3) *and* (3.4) *hold. Then any regular of system* (1.1)–(1.4) *satisfies*

$$E(t) \leqslant Ce^{-\xi t} E(0), \quad \forall t \geqslant 0,$$

where C *and* ξ *are positive constants.*

Proof. Using Lemmas 3.3 and 3.4 we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leqslant{}& -2N\alpha\|u_t\|^2_{L^2(\Omega_t)} - C_1 N g_1 \Box \nabla u + \|u_t\|^2_{L^2(\Omega_t)} \\
&- 2N\alpha\|v_t\|^2_{L^2(\Omega_t)} - C_1 N g_2 \Box \nabla v + \|v_t\|^2_{L^2(\Omega_t)} \\
&- \|\nabla u\|^2_{L^2(\Omega_t)} + \left(\int_0^t g_1(s)\,ds\right)\|\nabla u\|^2_{L^2(\Omega_t)} \\
&- \|\nabla v\|^2_{L^2(\Omega_t)} + \left(\int_0^t g_2(s)\,ds\right)\|\nabla v\|^2_{L^2(\Omega_t)} \\
&+ \|\nabla u\|^2_{L^2(\Omega_t)}\left(\int_0^t g_1(s)\,ds\right)^{\frac{1}{2}}(g \Box \nabla u)^{\frac{1}{2}} \\
&+ \|\nabla v\|^2_{L^2(\Omega_t)}\left(\int_0^t g_2(s)\,ds\right)^{\frac{1}{2}}(g_2 \Box \nabla v)^{\frac{1}{2}} \\
&- (2+\delta)\int_{\Omega_t} H(u-v)\,dx.
\end{aligned}$$

Using Young inequality we obtain for $\epsilon > 0$,

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leqslant{}& -2N\alpha\|u_t\|^2_{L^2(\Omega_t)} - C_1 N g_1 \Box \nabla u + \|u_t\|^2_{L^2(\Omega_t)} \\
&- \|\nabla u\|^2_{L^2(\Omega_t)} + \left(\int_0^t g_1(s)\,ds\right)\|\nabla u\|^2_{L^2(\Omega_t)} \\
&+ \frac{\epsilon}{2}\|\nabla u\|^2_{L^2(\Omega_t)} + \frac{\|g_1\|_{L^1(0,\infty)}}{2\epsilon} g_1 \Box \nabla u \\
&- 2N\alpha\|v_t\|^2_{L^2(\Omega_t)} - C_2 N g_2 \Box \nabla v + \|v_t\|^2_{L^2(\Omega_t)} \\
&- \|\nabla v\|^2_{L^2(\Omega_t)} + \left(\int_0^t g_2(s)\,ds\right)\|\nabla v\|^2_{L^2(\Omega_t)} \\
&+ \frac{\epsilon}{2}\|\nabla v\|^2_{L^2(\Omega_t)} + \frac{\|g_2\|_{L^1(0,\infty)}}{2\epsilon} g_2 \Box \nabla v
\end{aligned}$$

$$- (2+\delta) \int_{\Omega_t} H(u-v)\,dx.$$

Choosing N large enough and ϵ small we obtain

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -\lambda_0 E(t), \tag{3.8}$$

where λ_0 is a positive constant independent of t. From (3.7) and (3.8) it follows that

$$\mathcal{L}(t) \leqslant \mathcal{L}(0)e^{-\frac{\lambda_0}{k_1}t}, \quad \forall t \geqslant 0.$$

From equivalence relation (3.7) our conclusion follows. $\square$

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